On the structure of the adjacency matrix of the line digraph of a regular digraph

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Abstract

We show that the adjacency matrix M of the line digraph of a d-regular digraph D on n vertices can be written as $M = AB$, where the matrix A is the Kronecker product of the all-ones matrix of dimension d with the identity matrix of dimension n and the matrix B is the direct sum of the adjacency matrices of the factors in a dicycle factorization of D.

Key words: Line digraph; adjacency matrix; de Bruijn digraph

Introduction

Line digraphs of regular digraphs and their generalizations are important in the design of point-to-point interconnection networks for parallel computers and distributed systems. For instance, de Bruijn digraphs and Reddy-Pradhan-Kuhl digraphs, which are important topologies for interconnection networks, are all examples of line digraphs of regular digraphs (see, *e.g.*, [1],[2] and [6]). In this note, we describe a special regularity property of the adjacency matrix of the line digraph of a regular digraph. Before stating formally our main result, we recall the necessary graph-theoretic terminology.

A (*finite*) *directed graph*, for short *digraph*, consists of a non-empty finite set of elements called *vertices* and a (possibly empty) finite set of ordered pairs of vertices called *arcs*. The digraphs considered here are without multiple arcs. We denote by $D = (V, A)$ a digraph with vertex-set $V(D)$ and arc-set $A(D)$. A *labeling* of the vertices of a digraph D is a function $l : V(D) \longrightarrow L$, where L is a set of labels. Chosen a bijective labeling, the *adjacency matrix* of a digraph D with n vertices, denoted by $M(D)$, is the $n \times n$ $(0,1)$-matrix with ij-th element defined by $M_{i,j}(D) = 1$ if $(v_i, v_j) \in A(D)$ and $M_{i,j}(D) = 0$, otherwise. For any vertex $v_i \in V(D)$ of a digraph D, let $d^-_D(v_i) := |\{v_j : (v_j, v_i) \in A(D)\}|$

and $d_D^+(v_i) := |\{v_j : (v_i, v_j) \in A(D)\}|$. A digraph D is said to be *d-regular* if, for every vertex $v_i \in V(D)$, $d_D^-(v_i) = d_D^+(v_i) = d$. A digraph H is a *subdigraph* of a digraph D if $V(H) \subseteq V(D)$ and $A(H) \subseteq A(D)$. A subdigraph H of a digraph D is said to be a *spanning subdigraph* of D, or equivalently, a *factor* of D, if $V(H) = V(D)$. A *decomposition* of a digraph D is a set $\{H_1, H_2, ..., H_k\}$ of subdigraphs of D whose arc-sets are exactly the classes of a partition of $A(D)$. A *factorization* of a digraph D, if there exists one, is a decomposition of D into factors. A *dicycle factor* H of a digraph D is a spanning subdigraph of D such that $M(H)$ is a permutation matrix. The *disjoint union* of digraphs $D_1, D_2, ..., D_k$, is the digraph with vertex-set $\biguplus_{i=1}^k V(D_i)$, and arc-set $\biguplus_{i=1}^k A(D_i)$. Then a dicycle factor H of a digraph D is a spanning subdigraph of D and it is the disjoint union of dicycles. A *dicycle factorization* is a factorization into dicycle factors. The *line digraph* of a digraph D, denoted by $\overrightarrow{L}D$, is defined as follows: the vertex-set of $\overrightarrow{L}D$ is $A(D)$; for $v_h, v_i, v_j, v_k \in V(D)$, $((v_h, v_i), (v_j, v_k)) \in A(\overrightarrow{L}D)$ if and only if $v_i = v_j$. Kronecker product and direct sum of matrices M and N are respectively denoted by $M \otimes N$ and $M \oplus N$. The identity matrix and the all-ones matrix of size n are respectively denoted by I_n and J_n. In the next section, we prove the following theorem:

Theorem *Let D be a d-regular digraph on n vertices and let $\{H_1, H_2, ..., H_d\}$ be a dicycle factorization of D. Then there is a labeling of $V(\overrightarrow{L}D)$ such that*

$$M(\overrightarrow{L}D) = (J_d \otimes I_n) \bigoplus_{i=1}^{d} M(H_i).$$

1 Proof of the theorem

The proof of the theorem is based on two simple observations and a result proved by Hasunuma and Shibata [4] (see also Kawai *et al.* [5]).

Lemma 1 *Let D be a d-regular digraph. Then D has a dicycle factorization. In particular, if $\{H_1, H_2, ..., H_d\}$ is a dicycle factorization of D then $M(H_1), M(H_2), ..., M(H_d)$ are permutation matrices such that*

$$M(D) = \sum_{i=1}^{d} M(H_i).$$

Two digraphs D and D' are said to be *isomorphic* if there is a permutation matrix P such that $P \cdot M(D) \cdot P^{-1} = M(D')$. If D and D' are isomorphic we then write $D \cong D'$. An *n-dicycle*, denoted by $\overrightarrow{C}_n$, is a digraph with vertex-set

$\{v_1, v_2, ..., v_n\}$ and arc-set $\{(v_1, v_2), ..., (v_{n-1}, v_n), (v_n, v_1)\}$. A *$d$-spiked n-dicycle* is the digraph obtained from $\overrightarrow{C}_n$ as follows: for every vertex $v_i \in V(\overrightarrow{C}_n)$, we add d new vertices $w_1, w_2, ..., w_d$; we connect $v_i \in (\overrightarrow{C}_n)$ to the vertices $w_1, w_2, ..., w_d$, obtaining the arcs $(v_i, w_1), (v_i, w_2), ..., (v_i, w_d)$.

Lemma 2 *Let D be a d-spiked n-dicycle. Then $D \cong \overrightarrow{L} D$.*

Let D be a digraph and let H be a spanning subdigraph of D. The *growth* of D derived by H is the digraph denoted by $\Upsilon_D(H)$ and defined as follows: for every pair of vertices $v_i, v_j \in V(D)$, if $(v_i, v_j) \in A(H)$ then $(v_i, v_j) \in A(\Upsilon_D(H))$; for every vertex $v_i \in V(D)$, we add new vertices $w_1, w_2, ..., w_l$, where $l = d_D^+(v_i) - d_H^+(v_i)$; we connect $v_i \in V(D)$ to the vertices $w_1, w_2, ..., w_l$, obtaining the arcs $(v_i, w_1), (v_i, w_2), ..., (v_i, w_l)$.

Lemma 3 ([4]) *If $\{H_1, H_2, ..., H_k\}$ is a decomposition of a digraph D then*

$$\{\overrightarrow{L}\Upsilon_D(H_1), \overrightarrow{L}\Upsilon_D(H_2), ..., \overrightarrow{L}\Upsilon_D(H_k)\}$$

is a decomposition of a digraph $D' \cong \overrightarrow{L} D$.

Proof. [Proof of the theorem] Let D be a d-regular digraph on n vertices $v_1, v_2, ..., v_n$. Let $\{H_1, H_2, ..., H_d\}$ be a dicycle factorization of D. The vertices of $H_j \in \{H_1, H_2, ..., H_d\}$ are denoted as $(H_j, v_1), (H_j, v_2), ..., (H_j, v_n)$. Let us construct $\Upsilon_D(H_j)$. For every vertex $(H_j, v_i) \in V(H_j)$, we add $d-1$ new vertices to H_j. We label these new vertices by pairs of the form (H_l, v_m), for all $l \neq j$ and v_m such that $(v_i, v_m) \in A(H_l)$. In addition, $((H_j, v_i), (H_l, v_m)) \in A(\Upsilon_D(H_j))$. The digraph $\Upsilon_D(H_j)$ has $n \cdot d$ vertices. If we label the row number $(j-1)n + i$ of $M(\Upsilon_D(H_j))$ by the vertex (H_j, v_i), the adjacency matrix of $\Upsilon_D(H_j)$ is the $(d \cdot n) \times (d \cdot n)$ block-matrix

$$M(\Upsilon_D(H_j)) = \begin{pmatrix} \mathbf{0} \\ X_j \\ \mathbf{0} \end{pmatrix},$$

where

$$X_j = \begin{pmatrix} M(H_1) & M(H_2) & \cdots & M(H_j) & \cdots & M(H_{d-1}) & M(H_d) \end{pmatrix}.$$

Notice that $M(H_j)$ is the jj-th block of $M(\Upsilon_D(H_j))$. Thus, we have

$$
N=\sum_{i=j}^{d} M(\Upsilon_D(H_i)) = \begin{pmatrix} M(H_1) & M(H_2) & \cdots & M(H_d) \\ M(H_1) & M(H_2) & \cdots & M(H_d) \\ \vdots & \vdots & \ddots & \vdots \\ M(H_1) & M(H_2) & \cdots & M(H_d) \end{pmatrix}
$$

$$
= (J_d \otimes I_n) \bigoplus_{i=j}^{d} M(H_j).
$$

Observe that, for every $1 \leq j \leq d$, $\Upsilon_D(H_j)$ is the disjoint union of the d-spiked cycles corresponding to the orbits of the permutation associated to H_j. It follows from Lemma 2 that, for every $1 \leq j \leq d$,

$$
\Upsilon_D(H_j) \cong \overrightarrow{L}\Upsilon_D(H_j).
$$

Then, for the chosen labeling,

$$
M(\Upsilon_D(H_j)) = M(\overrightarrow{L}\Upsilon_D(H_j))
$$

and

$$
N = \sum_{j=1}^{d} M(\Upsilon_D(H_j)) = \sum_{j=1}^{d} M(\overrightarrow{L}\Upsilon_D(H_j)).
$$

Now, by Lemma 3, $N = M(\overrightarrow{L}D)$. ■

Remark The graph operation transforming a digraph D in its line digraph can be naturally iterated: $\overrightarrow{L}^k D := \overrightarrow{L}\,\overrightarrow{L}^{k-1}D$. Let Σ be an alphabet of cardinality d and let Σ^k be the set of all the words of length k over Σ. The *d-ary k-dimensional de Bruijn digraph*, denoted by $B(d,k)$, is defined as follows: the vertex-set of $B(d,k)$ is $V(B(d,k)) = \Sigma^k$; for every pair of vertices v_i, v_j, we have $(v_i, v_j) \in A(B(d,k))$ if and only if the last $k-1$ letters of v_i are the same as the first $k-1$ letters of v_j. Let K_d^+ be the complete digraph on d vertices with a loop at each vertex. Fiol, Yebra and Alegre [3] proved that $B(d,k) \cong \overrightarrow{L}^{k-1}K_d^+$. This result, together with the theorem, gives

$$
M(B(d,2)) \cong (J_d \otimes I_d) \bigoplus_{i=1}^{d} M(H_i),
$$

where $\{H_1, H_2, ..., H_d\}$ is any dicycle factorization of K_d^+.

Acknowledgments The author would like to thank the anonymous referees for helpful comments that improved this paper.

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